Integral Equation for Simultaneous Diagonalization of Two Covariance Kernels

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(Manuscript received January 20, 1967)

Let $K_1(s,t)$ and $K_2(s,t)$, $-T \leqq s,\ t \leqq T$, be real, symmetric, continuous and strictly positive-definite kernels, and denote by K_1 and K_2 the corresponding integral operators. Let $x(t)$ be a sample function of either of two zero-mean processes with covariances $K_1(s,t)$ and $K_2(s,t)$. We prove a generalized version of the following: If the integral equation

$$(K_2\psi_i)(t) = \lambda_i(K_1\psi_i)(t), \qquad -T \leqq t \leqq T,$$

has formal solutions λ_i and $\psi_i(t)$ which may contain δ-functions, and if $\{K_1\psi_i\}$ forms a complete set in $\mathfrak{L}_2[-T,T]$, then (i) the two kernels have the following simultaneous diagonalization:

$$K_1(s,t) = \sum_i (K_1\psi_i)(s)(K_1\psi_i)(t),$$

$$K_2(s,t) = \sum_i \lambda_i(K_1\psi_i)(s)(K_1\psi_i)(t),$$

uniformly on $[-T,T] \times [-T,T]$, and (ii) the sample function has an expansion

$$x(t) = \sum_i (x,\psi_i)(K_1\psi_i)(t)$$

in the stochastic mean, uniformly in t, and the coefficients are simultaneously orthogonal, i.e.,

$$E_1\{(x,\psi_i)(x,\psi_j)\} = \delta_{ij}\ , \qquad E_2\{(x,\psi_i)(x,\psi_j)\} = \lambda_i\ \delta_{ij}\ ,$$

where (x,ψ_i) is obtained by formally integrating $\psi_i(t)$ against $x(t)$.

I. INTRODUCTION

Let $K_1(s,t)$ and $K_2(s,t)$, $-T \leqq s,\ t \leqq T$, be real, symmetric, continuous and strictly positive-definite kernels, and denote by K_1 and

K_2 the integral operators with kernels $K_1(s,t)$ and $K_2(s,t)$. We have previously[1] established that, if $K_1^{-\frac{1}{2}}K_2K_1^{-\frac{1}{2}}$ is a densely defined and bounded operator on $\mathfrak{L}_2$ (the space of all square-integrable functions on $[-T,T]$) and if its extension to the whole of $\mathfrak{L}_2$ has eigenvalues λ_i and complete orthonormal eigenfunctions $\varphi_i(t)$, $i = 0, 1, \cdots$, then the two kernels have the following simultaneous diagonalization:

$$\begin{aligned} K_1(s,t) &= \sum_i (K_1^{\frac{1}{2}}\varphi_i)(s)(K_1^{\frac{1}{2}}\varphi_i)(t), \\ K_2(s,t) &= \sum_i \lambda_i(K_1^{\frac{1}{2}}\varphi_i)(s)(K_1^{\frac{1}{2}}\varphi_i)(t) \end{aligned} \tag{1}$$

uniformly on $[-T,T] \times [-T,T]$. In addition, if $x(t)$ is a sample function of either of two (separable and measurable) zero-mean processes with covariances $K_1(s,t)$ and $K_2(s,t)$ with associated measures P_1 and P_2, then

$$x(t) = \sum_i \eta_i(x)(K_1^{\frac{1}{2}}\varphi_i)(t) \tag{2}$$

in the stochastic mean, uniformly in t. Moreover,*

$$E_1\{\eta_i(x)\eta_j(x)\} = \delta_{ij}, \qquad E_2\{\eta_i(x)\eta_j(x)\} = \lambda_i\,\delta_{ij},$$

where†

$$\eta_i(x) = \lim_{n\to\infty} (x,K_1^{-\frac{1}{2}}\varphi_{in}) \tag{3}$$

in the stochastic mean, and $\{\varphi_{in}\}$ is any sequence of functions in the domain of $K_1^{-\frac{1}{2}}$ such that $\lim \| \varphi_i - \varphi_{in} \| = 0$.[1,2] Furthermore, if the two kernels have continuous $2r$th derivatives $(\partial^{2r}/\partial s^r\partial t^r)K_p(s,t)$, $p = 1, 2$, then (1) and (2) can be differentiated term-by-term r times while retaining the same senses of convergence.[1]

We remarked in Ref. 1 that, if φ_i is in the domain of $K_1^{-\frac{1}{2}}$, $\psi_i = K_1^{-\frac{1}{2}}\varphi_i$ satisfies the integral equation

$$(K_2\psi_i)(t) = \lambda_i(K_1\psi_i)(t), \qquad -T \leqq t \leqq T, \tag{4}$$

and

$$\begin{aligned} \eta_i(x) &= (x,\psi_i) \quad \text{a.s. (almost surely)}, \\ (K_1^{\frac{1}{2}}\varphi_i)(t) &= (K_1\psi_i)(t). \end{aligned} \tag{5}$$

Slepian (private communication) has long conjectured that, if (4) admits formal solutions λ_i and ψ_i, $i = 0, 1, \cdots$, where ψ_i may contain

* E_p, $p = 1, 2$, denotes the expectation with respect to P_p.

† For any f, $g \in \mathfrak{L}_2$, (f,g) denotes the inner product of f and g, and $\| f \|$ the norm of f.

δ-functions and their derivatives, then the expansion coefficients and functions of (2) are given by formally substituting such ψ_i into (5).* This conjecture, proved here, is significant since it provides a concrete means of obtaining the expansions (1) and (2). To illustrate the point, consider the following pair of covariance kernels:

$$K_1(s,t) = e^{-\alpha|s-t|}, \qquad K_2(s,t) = e^{-\beta|s-t|}.$$

For this pair, (4) admits the following formal solutions[5]

$$\begin{aligned} \bar{\psi}_{2k}(t) &= \cos\theta_k t + \frac{\cos\theta_k T}{\alpha+\beta}[\delta(t-T)+\delta(t+T)], \\ & \qquad\qquad\qquad\qquad k = 0, 1, \cdots, \\ \bar{\psi}_{2k+1}(t) &= \sin\hat{\theta}_k t + \frac{\sin\hat{\theta}_k T}{\alpha+\beta}[\delta(t-T)-\delta(t+T)], \end{aligned} \tag{6}$$

corresponding to

$$\lambda_{2k} = \frac{\beta}{\alpha}\frac{\alpha^2+\theta_k^2}{\beta^2+\theta_k^2}, \qquad \lambda_{2k+1} = \frac{\beta}{\alpha}\frac{\alpha^2+\hat{\theta}_k^2}{\beta^2+\hat{\theta}_k^2},$$

where θ_k and $\hat{\theta}_k$ are positivo solutions of

$$\begin{aligned} (\alpha+\beta)\theta_k \tan\theta_k T &= \alpha\beta - \theta_k^2, \\ -(\alpha+\beta)\hat{\theta}_k \operatorname{ctn}\hat{\theta}_k T &= \alpha\beta - \hat{\theta}_k^2, \end{aligned} \tag{8}$$

respectively, indexed in ascending order. Thus, formally,

$$\begin{aligned} (x,\bar{\psi}_{2k}) &= \int_{-T}^{T} x(t)\cos\theta_k t\,dt + \frac{\cos\theta_k T}{\alpha+\beta}[x(T)+x(-T)], \\ (x,\bar{\psi}_{2k+1}) &= \int_{-T}^{T} x(t)\sin\hat{\theta}_k t\,dt + \frac{\sin\hat{\theta}_k T}{\alpha+\beta}[x(T)-x(-T)], \end{aligned} \tag{9}$$

$$\begin{aligned} (K_1\bar{\psi}_{2k})(t) &= \frac{2\alpha}{\alpha^2+\theta_k^2}\cos\theta_k t, \\ (K_2\bar{\psi}_{2k+1})(t) &= \frac{2\alpha}{\alpha^2+\hat{\theta}_k^2}\sin\hat{\theta}_k t. \end{aligned} \tag{10}$$

Through a direct calculation, we previously[5] established that

(*i*) $K_1^{-\frac{1}{2}}K_2K_1^{-\frac{1}{2}}$ is densely defined and bounded,

(*ii*) its extension has eigenvalues λ_i given by (7) and complete

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* Similar conjectures have been made elsewhere.[3,4]

orthonormal eigenfunctions φ_i given as

$$\varphi_i = c_i \text{ l.i.m.} \sum_{j=0}^{n} \mu_{1j}^{\frac{1}{2}}(\psi_i , f_{1j})f_{1j} ,$$

(iii) $\eta_i = c_i(x,\tilde{\psi}_i)$ a.s.,* $K_1^{\frac{1}{2}}\varphi_i = c_i K_1 \tilde{\psi}_i$, which verifies Slepian's conjecture for this example. Here c_i is a normalization constant given by

$$c_{2k} = \left[\frac{2\alpha}{\alpha^2 + \theta_k^2}\left(T + \frac{(\alpha + \beta)\alpha\beta}{\theta_k^4 + (\alpha^2 + \beta^2)\theta_k^2 + \alpha^2\beta^2}\right)\right]^{-\frac{1}{2}},$$

$$c_{2k+1} = c_{2k} \big|_{\theta_k = \hat{\theta}_k} ,$$

(that is, c_{2k+1} is obtained by replacing θ_k with $\hat{\theta}_k$ in c_{2k}), μ_{pj} and f_{pj}, $p = 1, 2, j = 0, 1, \cdots$, are the eigenvalues and orthonormal eigenfunctions of K_p, and (ψ_i , f_{1j}) is defined analogously to (9).

In this paper we prove the generalization of (*i*), (*ii*), and (*iii*), starting with abstract kernels $K_1(s,t)$ and $K_2(s,t)$ and a generalized version of the integral equation (4).

II. MAIN RESULT

Theorem: Let $K_p(s,t)$, $p = 1, 2, -T \leqq s, t \leqq T$, be real, symmetric, strictly positive-definite kernels with continuous 2rth derivatives $(\partial^{2r}/\partial s^r \partial t^r)K_p(s,t)$. If there exist sequences of real numbers $\{a_{ilm}\}$, $\{t_m\}: -T \leqq t_m \leqq T$, and $\{\lambda_i\}$:

$$0 < b_1 \leqq \lambda_i \leqq b_2 , \qquad i = 0, 1, \cdots , \tag{11}$$

for some constants b_1 and b_2, and sequences of square-integrable functions $\{\psi_{il}\}$, which satisfy the equation

$$\begin{aligned} \sum_{l=0}^{r} \left[\int_{-T}^{T} \left(\frac{\partial^l}{\partial t^l} K_2(s,t)\right)\psi_{il}(t)\, dt + \sum_{m=1}^{q} a_{ilm} \frac{\partial^l}{\partial t^l} K_2(s,t) \bigg|_{t=t_m}\right] \\ = \lambda_i \sum_{l=0}^{r} \left[\int_{-T}^{T} \left(\frac{\partial^l}{\partial t^l} K_1(s,t)\right)\psi_{il}(t)\, dt + \sum_{m=1}^{q} a_{ilm} \frac{\partial^l}{\partial t^l} K_1(s,t) \bigg|_{t=t_m}\right], \\ -T \leqq s \leqq T, \end{aligned} \tag{12}$$

such that the right-hand side of (12) forms a complete set in $\mathfrak{L}_2$, then

(i) $K_1^{-\frac{1}{2}}K_2K_1^{-\frac{1}{2}}$ is a densely defined and bounded operator on $\mathfrak{L}_2$,

(ii) its extension to the whole of $\mathfrak{L}_2$ has eigenvalues and complete orthonormal eigenfunctions, which are the λ_i and

$$\varphi_i(s) = \sum_{l=0}^{r} \left[(K_{10l}^{\frac{1}{2}}\psi_{il})(s) + \sum_{m=1}^{q} a_{ilm}K_{10l}^{\frac{1}{2}}(s,t_m)\right], \tag{13}$$

* This portion is proved in a separate article.[6]

(iii) η_i and $K_1^{\frac{1}{2}}\varphi_i$ of (2) can be given, respectively, by

$$\eta_i(x) = \sum_{l=0}^{r} \left[(x^{(l)}, \psi_{il}) + \sum_{m=1}^{q} a_{ilm} x^{(l)}(t_m) \right] \quad \text{a.s.} \tag{14}$$

and by the right-hand side of (12) without λ_i. Here, $K_{p0l}^{\frac{1}{2}}$, $p = 1, 2$, denotes an integral operator whose kernel is defined as

$$K_{p0l}^{\frac{1}{2}}(s,t) = \sum_j \mu_{pj}^{\frac{1}{2}} f_{pj}(s) f_{pj}^{(l)}(t) \qquad l = 0, 1, \cdots, r, \tag{15}$$

in the mean in s, uniformly in t.

Remarks:

(*i*) $K_{p0l}^{\frac{1}{2}}(s,t)$ of (15) is well defined since

$$\sum_{j=0}^{\infty} \mu_{pj} f_{pj}^{(k)}(s) f_{pj}^{(l)}(t) = \frac{\partial^{k+l}}{\partial s^k \, \partial t^l} K_p(s,t), \qquad p = 1, 2, \tag{16}$$

uniformly in (s,t).[7] It follows from this that (15) converges in the mean in (s,t) as well. Hence, from Fubini's theorem, $K_{10l}^{\frac{1}{2}}(s,t)$ is a square-integrable function of t for almost every s. Thus, $\varphi_i(s)$ of (13) is well defined. We assume without loss of generality that φ_i, $i = 0, 1, \cdots$, are normalized.

(*ii*) For the example in Section I, $r = 0$, $q = 2$, $t_1 = T$, $t_2 = -T$, and

$$\psi_{2k,0}(t) = c_{2k} \cos \theta_k t, \qquad \psi_{2k+1,0}(t) = c_{2k+1} \sin \hat{\theta}_k t,$$

$$a_{2k,0,1} = a_{2k,0,2} = c_{2k} \frac{\cos \theta_k T}{\alpha + \beta},$$

$$a_{2k+1,0,1} = -a_{2k+1,0,2} = c_{2k+1} \frac{\sin \hat{\theta}_k T}{\alpha + \beta},$$

$$b_1 = \frac{\alpha}{\beta}, \qquad b_2 = \frac{\beta}{\alpha},$$

the right-hand side of (12) without λ_i is given by (10), and completeness of $\{\cos \theta_k t, \sin \hat{\theta}_k t\}$ follows from (18) and a gap-and-density theorem.[8]

III. PROOF OF THEOREM

For notational simplicity, we write K_{pkl}, $p = 1, 2$, for the integral operator whose kernel is

$$K_{pkl}(u,v) = \frac{\partial^{k+l}}{\partial u^k \, \partial v^l} K_p(u,v), \qquad k, l = 0, 1, \cdots, r.$$

K_{p00} and $K_{p00}^{\frac{1}{2}}$ are abbreviated as before by K_p and $K_p^{\frac{1}{2}}$, respectively.

(*i*) For any $f, g \in \mathfrak{L}_2$,

$$(K^{\frac{1}{2}}_{p0k}f, K^{\frac{1}{2}}_{p0l}g) = (f, K_{pkl}g), \tag{17}$$

$$K^{\frac{1}{2}}_{p0l}g = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} \mu_{pj}^{\frac{1}{2}} f_{pj}(f_{pj}^{(l)}, g). \tag{18}$$

To prove (17), note

$$\begin{aligned}(K^{\frac{1}{2}}_{p0k}f, K^{\frac{1}{2}}_{p0l}g) &= \iiint_{-T}^{T} f(s)g(t)K^{\frac{1}{2}}_{p0k}(u,s)K^{\frac{1}{2}}_{p0l}(u,t)\, ds\, dt\, du \\ &= \int_{-T}^{T}\int f(s)g(t) \sum_{i} \mu_{pi} f_{pi}^{(k)}(s) f_{pi}^{(l)}(t)\, ds\, dt \\ &= (f, K_{pkk}g),\end{aligned}$$

where the second equality follows from the mean convergence of (15) and the third from the uniform convergence of (16). To prove (18), consider

$$\left\| K^{\frac{1}{2}}_{p0l}g - \sum_{j=0}^{n} \mu_{pj}^{\frac{1}{2}} f_{pj}(f_{pj}^{(l)}, g) \right\|^2 = (g, K_{pll}g) - \sum_{j=0}^{n} \mu_{pj}(f_{pj}^{(l)}, g)^2,$$

which vanishes as $n \to \infty$ since (16) converges uniformly in (s,t).

(*ii*) $K_2^{-\frac{1}{2}}K_1^{\frac{1}{2}}$ and $K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}$ are densely defined and bounded on $\mathfrak{L}_2$.

To prove this, apply $K_2^{-\frac{1}{2}}$ on both sides of (12) and use (18) to obtain

$$\sum_{l=0}^{r}\left[K^{\frac{1}{2}}_{20l}\psi_{il} + \sum_{m=1}^{q} a_{ilm}K^{\frac{1}{2}}_{20l}(\cdot, t_m)\right] = \lambda_i K_2^{-\frac{1}{2}}K_1^{\frac{1}{2}}\varphi_i .$$

Then, for each i,

$$\begin{aligned}&\lambda_i^2 \,\| K_2^{-\frac{1}{2}}K_1^{\frac{1}{2}}\varphi_i \|^2 \\ &\quad= \sum_{k,l=0}^{r}\Big\{(K^{\frac{1}{2}}_{20k}\psi_{ik}, K^{\frac{1}{2}}_{20l}\psi_{il}) \\ &\qquad + \sum_{m=1}^{q}[a_{ilm}(K^{\frac{1}{2}}_{20l}(\cdot,t_m), K^{\frac{1}{2}}_{20k}\psi_{ik}) + a_{ikm}(K^{\frac{1}{2}}_{20k}(\cdot,t_m), K^{\frac{1}{2}}_{20l}\psi_{il})] \\ &\qquad + \sum_{m,n=1}^{q} a_{ilm}a_{ikn}(K^{\frac{1}{2}}_{20l}(\cdot,t_m), K^{\frac{1}{2}}_{20k}(\cdot,t_n))\Big\} \\ &\quad= \sum_{k,l=0}^{r}\Big\{\Big(\psi_{ik}, K_{2kl}\psi_{il} + \sum_{m=1}^{q} a_{ilm}K_{2kl}(\cdot,t_m)\Big) \\ &\qquad + \sum_{n=1}^{q} a_{ikn}\Big[(K_{2kl}\psi_{il})(t_n) + \sum_{m=1}^{q} a_{ilm}K_{2kl}(t_n, t_m)\Big]\Big\}\end{aligned}$$

$$= \lambda_i \sum_{k,l=0}^{r} \left[\left(K_{10k}^{\frac{1}{2}}\psi_{ik} , K_{10l}^{\frac{1}{2}}\psi_{il} + \sum_{m=1}^{q} a_{ilm} K_{10l}^{\frac{1}{2}}(\cdot, t_m) \right) \right.$$

$$\left. + \sum_{n=1}^{q} a_{ikn} \left(K_{10k}^{\frac{1}{2}}(\cdot, t_n), K_{10l}^{\frac{1}{2}}\psi_{il} + \sum_{m=1}^{q} a_{ilm} K_{10l}^{\frac{1}{2}}(\cdot, t_m) \right) \right]$$

$$= \lambda_i \, \| \varphi_i \|^2,$$

where the second equality follows from (17) and (18), the third from k time differentiation of (12) and from (17) and (18), and the last from (13). Hence, with φ_i being normalized,

$$\| K_2^{-\frac{1}{2}} K_1^{\frac{1}{2}} \varphi_i \|^2 = \frac{1}{\lambda_i}, \qquad i = 0, 1, \cdots .$$

Now $\{\varphi_i\}$ is complete since the right-hand side of (12) without λ_i, which forms a complete set by hypothesis, is equal to $K_1^{\frac{1}{2}}\varphi_i$, and $K_1^{\frac{1}{2}}$ is strictly positive-definite. Hence, from (11), $K_2^{-\frac{1}{2}}K_1^{\frac{1}{2}}$ is densely defined and bounded.

To prove that $K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}$ is also densely defined and bounded, define $\hat{\varphi}_i$ as the normalized right-hand side of (13) with the subscript 1 replaced by 2. Completeness of $\{\hat{\varphi}_i\}$ is similarly deduced via (12). Now, by following the same procedure with the roles of K_1 and K_2 interchanged, we obtain

$$\| K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} \hat{\varphi}_i \|^2 = \lambda_i , \qquad i = 0, 1, \cdots .$$

Then, the assertion follows immediately from (11).

(*iii*) The ranges of $K_1^{\frac{1}{2}}$ and $K_2^{\frac{1}{2}}$ are equal, namely,

$$K_1^{\frac{1}{2}}(\mathfrak{L}_2) = K_2^{\frac{1}{2}}(\mathfrak{L}_2).$$

To prove this, denote by L and M the extensions to the whole of $\mathfrak{L}_2$ of $K_2^{-\frac{1}{2}}K_1^{\frac{1}{2}}$ and $K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}$ respectively, which exist as a result of (*ii*). Since the domains of $K_2^{\frac{1}{2}}L$ and $K_1^{\frac{1}{2}}M$ are $\mathfrak{L}_2$, which is also the domains of $K_1^{\frac{1}{2}}$ and $K_2^{\frac{1}{2}}$, we have

$$K_1^{\frac{1}{2}} = K_2^{\frac{1}{2}}L, \qquad K_2^{\frac{1}{2}} = K_1^{\frac{1}{2}}M.$$

Then, from the first equality, $K_1^{\frac{1}{2}}(\mathfrak{L}_2) \subset K_2^{\frac{1}{2}}(\mathfrak{L}_2)$, while, from the second, $K_2^{\frac{1}{2}}(\mathfrak{L}_2) \subset K_1^{\frac{1}{2}}(\mathfrak{L}_2)$. Hence, the assertion holds.

(*iv*)

$$K_{20l}^{\frac{1}{2}}(\cdot, t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} K_2^{\frac{1}{2}} f_{1j} f_{1j}^{(l)}(t), \qquad -T \leqq t \leqq T, \tag{19}$$

$$K_{20l}^{\frac{1}{2}} g = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} K_2^{\frac{1}{2}} f_{1j} (f_{1j}^{(l)}, g), \qquad g \in \mathfrak{L}_2 . \tag{20}$$

To prove (19), note first that f_{1j}, $j = 0, 1, \cdots$, are in the domain of $K_2^{-\frac{1}{2}}$ as a result of (*iii*) and also that $(K_2^{-\frac{1}{2}}f_{1i}, K_2^{\frac{1}{2}}f_{1j}) = \delta_{ij}$ from orthonormality of $\{f_{1j}\}$. Thus, $\{K_2^{-\frac{1}{2}}f_{1j}\}$ and $\{K_2^{\frac{1}{2}}f_{1j}\}$ form a pair of mutually reciprocal bases of $\mathfrak{L}_2$. Hence,

$$K_{20t}^{\frac{1}{2}}(\cdot,t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} K_2^{\frac{1}{2}}f_{1j}(K_2^{-\frac{1}{2}}f_{1j}, K_{20t}^{\frac{1}{2}}(\cdot,t)). \tag{21}$$

But from (15)

$$(K_2^{-\frac{1}{2}}f_{1j}, K_{20t}^{\frac{1}{2}}(\cdot,t)) = \sum_{i=0}^{\infty} (f_{1j}, f_{2i})f_{2i}^{(l)}(t), \qquad l = 0, 1, \cdots, r, \tag{22}$$

uniformly in t. Now, since $\{f_{2i}\}$ is an orthonormal basis of $\mathfrak{L}_2$,

$$f_{1j} = \underset{n\to\infty}{\text{l.i.m.}} \sum_{i=0}^{n} (f_{1j}, f_{2i})f_{2i}.$$

But, according to (22), the right-hand side converges uniformly. Hence, the above partial sum must converge uniformly to f_{1j}. Suppose for some k, $0 \leqq k < r$,

$$f_{1j}^{(k)}(t) = \sum_{i=0}^{\infty} (f_{1j}, f_{2i})f_{2i}^{(k)}(t) \tag{23}$$

uniformly in t. Then, from (22),

$$f_{1j}^{(k+1)}(t) = \sum_{i=0}^{\infty} (f_{1j}, f_{2i})f_{2i}^{(k+1)}(t)$$

uniformly in t.[9] Hence, by induction, (23) holds for every k, $0 \leqq k \leqq r$. Therefore, from (22),

$$(K_2^{-\frac{1}{2}}f_{1j}, K_{20t}^{\frac{1}{2}}(\cdot,t)) = f_{1j}^{(l)}(t), \qquad l = 0, 1, \cdots, r.$$

Then, (19) follows from (21) and the above.

To prove (20), we expand $K_{20t}^{\frac{1}{2}}g$ relative to $\{K_2^{\frac{1}{2}}f_{1j}\}$:

$$K_{20t}^{\frac{1}{2}}g = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} (K_2^{-\frac{1}{2}}f_{1j}, K_{20t}^{\frac{1}{2}}g)K_2^{\frac{1}{2}}f_{1j},$$

and note from (18) and (23) that

$$(K_2^{-\frac{1}{2}}f_{1j}, K_{20t}^{\frac{1}{2}}g) = \sum_{i=0}^{\infty} (f_{1j}, f_{2i})(f_{2i}^{(l)}, g) = (f_{1j}^{(l)}, g).$$

(*v*) To prove (*i*) of the theorem, we note from (*ii*) and (*iii*) that $K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}$ is everywhere-defined and bounded on $\mathfrak{L}_2$. Hence, its adjoint $(K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}})^*$ is also everywhere-defined and bounded. Now, for any

$f \in \mathcal{L}_2$ and $g \in \mathfrak{D}(K_1^{-\frac{1}{2}})$, the domain of $K_1^{-\frac{1}{2}}$, we have $(K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}f,g) = (f,K_2^{\frac{1}{2}}K_1^{-\frac{1}{2}}g)$. Thus, $K_2^{\frac{1}{2}}K_1^{-\frac{1}{2}}g = (K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}})^*g$, $g \in \mathfrak{D}(K_1^{-\frac{1}{2}})$. Hence, $K_2^{\frac{1}{2}}K_1^{-\frac{1}{2}}$ is bounded. Since $\mathfrak{D}(K_1^{-\frac{1}{2}})$ is dense in $\mathcal{L}_2$, we conclude that $K_1^{-\frac{1}{2}}K_2K_1^{-\frac{1}{2}}$ is densely defined and bounded.

(*vi*) To prove (*ii*) of the theorem, define

$$\varphi_{in}(t) = \sum_{j=0}^{n} \mu_{1j}^{\frac{1}{2}} \sum_{l=0}^{r} \left[(\psi_{il}, f_{1j}^{(l)}) + \sum_{m=1}^{q} a_{ilm} f_{1j}^{(l)}(t_m) \right] f_{1j}(t), \tag{24}$$

and note $\varphi_{in} \in \mathfrak{D}(K_1^{-\frac{1}{2}})$ and $\lim_{n\to\infty} \| \varphi_i - \varphi_{in} \| = 0$. Then

$$\begin{aligned}
\underset{n\to\infty}{\text{l.i.m.}}\, K_2K_1^{-\frac{1}{2}}\varphi_{in} &= \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} \sum_{l=0}^{r} \left[(\psi_{il}, f_{1j}^{(l)}) + \sum_{m=1}^{q} a_{ilm} f_{1j}^{(l)}(t_m) \right] K_2 f_{1j} \\
&= \sum_{l=0}^{r} \left[K_{20l}\psi_{il} + \sum_{m=1}^{q} a_{ilm} K_{20l}(\cdot, t_m) \right] \\
&= \lambda_i K_1^{\frac{1}{2}} \varphi_i ,
\end{aligned}$$

where the second equality follows from (19), (20), (15) and (18), and third from (12) and (13). Now denote by Q the extension of $K_1^{-\frac{1}{2}}K_2K_1^{-\frac{1}{2}}$ to the whole of $\mathcal{L}_2$. Then,

$$K_1^{\frac{1}{2}}Qf = \underset{n\to\infty}{\text{l.i.m.}}\, K_2K_1^{-\frac{1}{2}}f_n$$

for any $f \in \mathcal{L}_2$ and $\{f_n\}: f_n \in \mathfrak{D}(K_1^{-\frac{1}{2}})$, $\lim \| f - f_n \| = 0$, since

$$\| K_1^{\frac{1}{2}}Qf - K_2K_1^{-\frac{1}{2}}f_n \| \leqq \| K_1^{\frac{1}{2}}Q(f - f_n) \| + \| (K_1^{\frac{1}{2}}Q - K_2K_1^{-\frac{1}{2}})f_n \|$$

which vanishes as $n \to \infty$. Therefore, $Q\varphi_i = \lambda_i\varphi_i$. Lastly, since $\{\varphi_i\}$ is complete in $\mathcal{L}_2$, $\{\lambda_i\}$ constitutes the entire spectrum of Q.

(*vii*) To prove (*iii*) of the theorem, note from (3), (24) and (*vi*) that

$$\eta_i(x) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=0}^{n} \sum_{l=0}^{r} \left[(\psi_{il}, f_{1j}^{(l)}) + \sum_{m=1}^{q} a_{ilm} f_{1j}^{(l)}(t_m) \right] (f_{1j}, x).$$

Now

$$\begin{aligned}
E_1 \left| (x^{(l)}, \psi_{il}) - \sum_{j=0}^{n} (x, f_{1j})(f_{1j}^{(l)}, \psi_{il}) \right|^2 \\
= (\psi_{il}, K_{1ll}\psi_{il}) - \sum_{j=0}^{n} \mu_{1j}(\psi_{il}, f_{1j}^{(l)})^2,
\end{aligned}$$

$$E_1 \left| x^{(l)}(t) - \sum_{j=0}^{n} (x, f_{1j}) f_{1j}^{(l)}(t) \right|^2 = K_{1ll}(t,t) - \sum_{j=0}^{n} \mu_{1j}[f_{1j}^{(l)}(t)]^2,$$

both of which vanish as $n \to \infty$ by virtue of (16). Also, with the use of (17) and (18),

$$E_2 \left| (x^{(l)},\psi_{il}) - \sum_{j=0}^{n} (x,f_{1j})(f_{1j}^{(l)},\psi_{il}) \right|^2$$

$$= (\psi_{il}\,,K_{2ll}\psi_{il}) - 2 \sum_{j=0}^{n} (\psi_{il}\,,f_{1j}^{(l)})(f_{1j}\,,K_{20l}\psi_{il})$$

$$+ \sum_{j,k=0}^{n} (\psi_{il}\,,f_{1j}^{(l)})(\psi_{il}\,,f_{1k}^{(l)})(f_{1j}\,,K_2 f_{1k})$$

$$= \left\| K_{20l}^{\frac{1}{2}}\psi_{il} - \sum_{j=0}^{n} K_2^{\frac{1}{2}} f_{1j}(f_{1j}^{(l)},\psi_{il}) \right\|^2,$$

$$E_2 \left| x^{(l)}(t) - \sum_{j=0}^{n} (x,f_{1j}) f_{1j}^{(l)}(t) \right|^2 = K_{2ll}(t,t) - 2 \sum_{j=0}^{n} f_{1j}^{(l)}(t)(K_{2l0}f_{1j})(t)$$

$$+ \sum_{j,k=0}^{n} f_{1j}^{(l)}(t) f_{1k}^{(l)}(t)(f_{1j}\,,K_2 f_{1k}) = \left\| K_{20l}^{\frac{1}{2}}(\cdot\,,t) - \sum_{j=0}^{n} K_2^{\frac{1}{2}} f_{1j} f_{1j}^{(l)}(t) \right\|^2,$$

both of which vanish as $n \to \infty$ by virtue of (19) and (20). Therefore, upon combination of the above results, (14) is proved.

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